Eur. Phys. J. C 27, 453–455 (2003)
Digital Object Identifier (DOI) 10.1140/epjc/s2002-01097-x

THE EUROPEAN
PHYSICAL JOURNAL C

Invariance quantum group of the fermionic oscillator

M. Arık[1], S. Gün[2], A. Yıldız[3]

[1] Department of Physics, Boğaziçi University, Bebek, İstanbul, Turkey
[2] Department of Physics, Istanbul University, Vezneciler, İstanbul, Turkey
[3] Feza Gürsey Institute, P.O. Box 6, 81220, Çengelköy, İstanbul, Turkey

Received: 15 July 2002 / Revised version: 14 October 2002 /
Published online: 19 February 2003 –

Abstract. The fermionic oscillator defined by the algebraic relations $cc^* + c^*c = 1$ and $c^2 = 0$ admits the homogeneous group $O(2)$ as its invariance group. We show that the structure of the inhomogeneus invariance group of this oscillator is a quantum group.

1 Introduction

Quantum field theory, which describes the ultimate behavior of elementary particles and fields in physics, fundamentally depends on the concepts of the bosonic oscillator described by the algebraic relation

$$aa^* - a^*a = 1 \tag{1}$$

and the fermionic oscillator described by the algebraic relations

$$cc^* + c^*c = 1, \tag{2}$$
$$c^2 = 0. \tag{3}$$

The algebra of the bosonic oscillator (1) is invariant under the inhomogeneous symplectic group $ISp(2, R)$ which transforms a, a^* and 1 into each other. The homogeneous part of this group which just transforms a, a^* into each other is $Sp(2, R) \simeq SU(1, 1)$. For the fermionic oscillator (2) we should like to remark the importance of the relation $c^2 = 0$. Fermions satisfy the Pauli exclusion principle; the two identical fermions cannot occupy the same state. Thus $c^2 = 0$. The algebra (2) describes the simplest non-trivial quantum mechanical system. In this sense it is most fundamental. Although the bosonic oscillator (1) has a classical limit in which it reduces to the harmonic oscillator, the fermionic oscillator (2) has no classical analogue. Thus a thorough understanding of all its properties is important. One important property of the algebra (2) is that it does not admit a q-deformation [1–4]. Another property is that although it is invariant under the orthogonal group $O(2)$ which transforms c and c^* into each other there is no inhomogeneous classical Lie group which transforms c, c^* and 1 into each other. In this paper we construct a quantum group [5–8] which achieves this purpose. We show that the structure of the inhomogeneous invariance group of the fermionic oscillator is a quantum group, that is, the matrix elements of the transformation matrix which transforms c, c^* and 1 into each other belong to a non-commutative Hopf algebra [5–8] where the co-product is given by the matrix product. We will develop the R-matrix formulation of this quantum group and show that the operators generating this quantum group have a two dimensional representation, which we explicitly construct. The representation matrices depend on five parameters. We finally present a discussion of our results.

To show that the structure of the inhomogeneous invariance "group" of the fermionic oscillator is a quantum group, we consider a 3×3 matrix A whose elements belong to an algebra $\mathcal{A}$. We form the column matrix

$$\mathbf{c} = \begin{bmatrix} c \\ c^* \\ 1 \end{bmatrix}, \tag{4}$$

and assume that the action of the matrix A on c is given by

$$\mathbf{c}' = A \overset{\bullet}{\otimes} \mathbf{c} \tag{5}$$

We assume that the matrix A is of the form

$$A = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

in accordance with the general form of inhomogeneous transformations of c and c^* so that the transformed fermion algebra generators in (4) are explicitly given by

$$c' = \alpha \otimes c + \beta \otimes c^* + \gamma \otimes 1, \tag{7}$$
$$c^{*'} = \alpha^* \otimes c^* + \beta^* \otimes c + \gamma^* \otimes 1. \tag{8}$$

If α, β, γ are taken as complex numbers the invariance of the fermion algebra (2) requires that $\alpha = \beta = 0$, $\alpha = \mathrm{e}^{\mathrm{i}\rho}$

or $\gamma = \alpha = 0$, $\beta = \mathrm{e}^{-\mathrm{i}\rho}$ giving the homogeneous group $O(2)$. If we assume that α, β, γ and their hermitian conjugates can form a non-commutative algebra, the conditions that $\mathbf{c}'$ satisfies the relations (2) give rise to 12 (real) relations. It is known that when some non-linear completely integrable systems are quantized the Lie group which describes the symmetries of the system has also to be quantized to yield a Hopf algebra. We thus look for a Hopf algebra structure for $\mathcal{A}$. In order to accomplish this, these 12 relations must be supplemented by five additional relations so that the 17 defining relations of the algebra $\mathcal{A}$ are given by

$$\alpha\alpha^* = \alpha^*\alpha, \tag{9}$$
$$\beta\beta^* = \beta^*\beta, \tag{10}$$
$$\gamma\gamma^* + \gamma^*\gamma = 1 - \alpha^*\alpha - \beta^*\beta, \tag{11}$$
$$\alpha\beta = \beta\alpha, \tag{12}$$
$$\alpha\beta^* = \beta^*\alpha, \tag{13}$$
$$\gamma^2 = -\alpha\beta, \tag{14}$$
$$\alpha\gamma = -\gamma\alpha, \tag{15}$$
$$\alpha\gamma^* = -\gamma^*\alpha, \tag{16}$$
$$\beta\gamma = -\gamma\beta, \tag{17}$$
$$\beta\gamma^* = -\gamma^*\beta, \tag{18}$$

plus the hermitian conjugates of (12)–(18). Note that α, α^*, β, β^*, 1 commute among themselves. For the special case $\alpha = \beta = 0$, γ, γ^* and 1 satisfy the fermion algebra. We find the Hopf algebra with the co-product given by matrix multiplication

$$\Delta(A) = \begin{bmatrix} \Delta(\alpha) & \Delta(\beta) & \Delta(\gamma) \\ \Delta(\beta^*) & \Delta(\alpha^*) & \Delta(\gamma^*) \\ 0 & 0 & 1 \end{bmatrix} = A \overset{\bullet}{\otimes} A, \tag{19}$$

the co-unit given by the unit matrix

$$\epsilon(A) = I, \tag{20}$$

and the antipode given by

$$S(A) = \delta^{-1} \begin{bmatrix} \alpha^* & -\beta & -\alpha^*\gamma + \beta\gamma^* \\ -\beta^* & \alpha & -\alpha\gamma^* + \beta^*\gamma \\ 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

where

$$\delta = \alpha\alpha^* - \beta\beta^* \tag{22}$$

is a central element of the algebra.

The fermionic oscillator algebra (2) and (3) can be written as a vector algebra

$$R\mathbf{C}_1\mathbf{C}_2 = C_2\mathbf{C}_1, \tag{23}$$

where

$$\mathbf{C}_1\mathbf{C}_2 = \begin{bmatrix} c^2 \\ cc^* \\ c \\ c^*c \\ (c^*)^2 \\ c^* \\ c \\ c^* \\ 1 \end{bmatrix}, \quad \mathbf{C}_2\mathbf{C}_1 = \begin{bmatrix} c^2 \\ c^*c \\ c \\ cc^* \\ (c^*)^2 \\ c^* \\ c \\ c^* \\ 1 \end{bmatrix}, \tag{24}$$

and the 9×9 R-matrix is

$$R = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{25}$$

The invariance of (23) under the transformation (5) implies that the matrix A satisfies

$$RA_1A_2 = A_2A_1R. \tag{26}$$

Since the R-matrix satisfies the quantum Yang–Baxter equation, the matrix A whose entries satisfy (9)–(18) defines a quantum matrix group.

The only irreducible representation of the fermion algebra (2) is two dimensional and can be written in terms of the Pauli matrices:

$$c = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) = \sigma_+,$$
$$c^* = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2) = \sigma_-. \tag{27}$$

The overall phase ρ can be identified with the familiar $SO(2)$ group acting on c by $c \to \mathrm{e}^{\mathrm{i}\rho}c$. Since for the special case $\alpha = \beta = 0$ representations of the algebra $\mathcal{A}$ given by (9)–(18) must reduce to the representations of the fermion algebra (2) we may deduce that, if representations of $\mathcal{A}$ depend on a number of parameters which take special values for the case $\alpha = \beta = 0$, then $\mathcal{A}$ can only have a two dimensional irreducible representation. This representation is given by

$$\alpha = \alpha_3\sigma_3,$$
$$\beta = \beta_3\sigma_3,$$
$$\gamma = \gamma_+\sigma_+ + \gamma_-\sigma_-, \tag{28}$$

where the complex numbers $\alpha_3, \beta_3, \gamma_+, \gamma_-$ are chosen such that (9)–(18) are satisfied;

$$|\alpha_3|^2 + |\beta_3|^2 + |\gamma_+|^2 + |\gamma_-|^2 = 1,$$
$$\gamma_+\gamma_- + \alpha_3\beta_3 = 0. \tag{29}$$

A particular parameterization is given by

$$\begin{aligned} \alpha_3 &= \mathrm{e}^{\mathrm{i}(\rho+\sigma)} \cos\theta \cos\varphi, \\ \beta_3 &= \mathrm{e}^{\mathrm{i}(\rho-\sigma)} \sin\theta \sin\varphi, \\ \gamma_+ &= \mathrm{e}^{\mathrm{i}(\rho+\tau)} \cos\theta \sin\varphi, \\ \gamma_- &= -\mathrm{e}^{\mathrm{i}(\rho-\tau)} \sin\theta \cos\varphi. \end{aligned} \tag{30}$$

Thus the fermionic inhomogeneous orthogonal quantum group $FIO(2)$ depends on five angles. The fact that the parameters are all angles as well as (11) shows that $FIO(2)$ is compact as compared to the bosonic invariance group $ISp(2)$ which has the same number of parameters but is non-compact. In contrast, the familiar inhomogeneous orthogonal group $IO(2)$, which is the Euclidean group in two dimensions, has three parameters.

In physics, symmetries are important. Fundamental examples are the Lorentz invariance of special relativity and the rotational invariance of hydrogen atom. Until the 1980's it was thought that when a physical system is quantized the classical group which describes the symmetries of the system remain intact. i.e. it remains a classical group. When some non-linear, completely integrable systems were quantized in the 80's it was discovered that the group which describes the symmetries of the physical system has also to be quantized, namely the classical group which acts on the classical system has to change into a quantum group. In the simplest examples, the classical matrix with commuting elements has to turn into a matrix with non-commuting elements. Moreover the algebra generated by these elements has to satisfy the axioms of a Hopf algebra. The quantum groups discovered in this fashion have the property that in some limit they reduce to a classical group. A fermionic system, on the other hand, does not have any classical analogue. By showing that the "inhomogeneous invariance group" of the fermionic oscillator is not a classical group but a quantum group, we have remotivated the introduction of quantum groups into field theory.

References

1. Ws. Chung, Int. J. Theor. Phys. **33**, 1611 (1994)
2. A. Solomon, R. McDermot, J. Phys. A Math. Gen. **27**, 2619 (1994)
3. Ws. Chung, Prog. Theor. Physics **95**, 3 (1996)
4. An-Min Wang, Si-Cong Jing, Tu-Nan Ruan, Il Nuovo Cimento A **107**, 9 (1994)
5. L.D. Faddeev, N.Y. Reshetikhin, L.A. Takhtajan, Algebraic Analysis **1**, 129 (1988)
6. M. Jimbo, Lett. Math. Phys. **11**, 247 (1986)
7. V.G. Drinfeld, Proceedings International Congr. Math., Berkeley **1**, 798 (1986)
8. S.L. Woronowicz, Commun. Math. Phys. **111**, 613 (1987)